The below message, entitled "Dissertation Re Despairing Drunkards", was written and circulated to Christian Fellowship members Saturday March 10, 2007

Greetings all in the Delightfully Delectable Name of JESUS!!

"There is a way which seemeth right unto a man, but the end thereof are the ways of death." Prov 14:12

This week marks the first anniversary of the launch of the Christian Fellowship in the Workplace (the Mount of Praise Sunday morning service that marked our launch took place on March 5, 2006).

I have been giving much thought and prayer over the last two weeks on the way forward towards revival in our Fellowships and in our land, and I thought I would write another "dissertation", this time on the fundamentally important topic of getting close to God. I am reminded in this regard of a song my late Grandmother used to love to sing (the older ones among those getting this email may know it!):

"Show me the way to go home,
I am tired and I want to go to bed,
I had a little drink about an hour ago,
And it's gone straight to my head
Where ever I may roam
On land or sea or foam
You will always hear me singing this song
Show me the way to go home."

My Grandmother, please note, was not an alcoholic, but she used to like singing the words of this old (1925)
Irving King song, and it turns out that the inability of the inebriated to locate his or her domicile is of far more than passing interest, both to scientists and to us.

In mathematics there is a very interesting theory (G. Poyla's 1921 Theorem) which states that the simple random (or Drunkard's) walk in one (1D) or two dimensions (2D) is "recurrent" (the walker will always return home), but in three (3D) or higher dimensions it is "transient" (the walker is likely to wander off for ever).

For example, in 1D, consider the intoxicated itinerant who, normally housed in the middle of a street, wanders away one night, weaving randomly up or down the road after each house reached. Let P be the probability of returning home, given that each door passed is knocked on, and given that the sleeping
occupants of the gap's two corner houses are policemen who will arrest the tiresome tottering traveler if roused from their beds by the knocking. If the only houses are home and the corner houses, then P is zero – the wobbly walker is certain to end up jailed. If there is an extra house each way between home and corner houses, the odds of the

despairing drunkard returning home will be 50-50 (P=1/2), and these odds will improve with the number of doors; for a road with an infinite number, homecoming is 100% certain (P=1), maybe in time for breakfast.

Similarly, an anxious alcoholic meandering the streets of an infinitely large city (the 2D random walk) and trying to find the way home is also certain eventually to succeed, though more likely to miss both breakfast and lunch.

In three dimensions, as Poyla's theorem indicates, the story is radically different; here one is more likely to get lost for good. E.g. if all the planets in our universe (assumed infinite in size) were at the nodes of a cubic grid, and a spaceship's carousing Commander was randomly navigating that grid having left Earth, the odds would be roughly 2 to 1 against the sloshed or "spaced-out" Star-Trekie sampling even supper ever again (P ~ 0.341).

The spiritual relevance of all this is our tendency to stray from God, from our calling, or from "Home"... i.e. where we should be spiritually. We are 3D creatures, and would get lost in the natural, but this our tendency to get "lost" is even more true when we add the spiritual dimension.

There seems to a law, like Poyla's, which is in operation to pull us away from heaven or "Home". Nor is it a figment of our imagination; the apostle Paul talks about it in Romans chapter 7, verse 23, as the "law in my members, warring against the law in my mind, and bringing me into captivity to the law of sin which is in my members" .

Of course, the sinner who knows nothing about God is the one who is truly lost, because not only is he or she straying from God, but the sad thing is that such a person does not even know there is a "Home" where he or she should be headed.

However, I am thinking here of us, the believers, who know very well there is a "Home" and yet stray.

Is sin, or impurities in our life, the problem? At first one would deem this the root cause of our not finding our way Home to God, but while this is a big reason we find it hard to get close to God, the problem is even deeper - it is our very nature which most hinders us. Once again one can model this using scientific analogy.

Scientists tell us that the motion of electrons involved in electrical conduction closely resembles a random walk, and that, in line with Poyla's theorem, a metal with impurities always becomes an insulator in 1 and 2D, whereas it can remain a good conductor in 3D - this is because the presence of impurities hinders but does not affect overall the electrons finding their way "home". Analysis shows that the electrical resistance between two points far apart in a lattice remains roughly the same despite the presence of impurities. The resistance to electricity near the impurity is increased (and the more so the nearer to the impurity, or the greater the impurity) but the overall behaviour of the lattice remains, electrically speaking, the same.

The spiritual application here is this: impurities in our lives are stumbling blocks between us and God, but our very nature (pride, self-righteousness, wanting our own way) is the bigger force, the greater resistance, at work blocking God's presence from reaching our hearts. Nor is our sinful nature the only force at work hindering us - there is also the pull of the enemy of
our souls trying to get us further and further away from God.

Here also a scientific analogy can be invoked to help us see the bigger picture - namely the pull of electrical voltage such as that provided by a battery. When a battery is placed across a metal that conducts electricity, the electrons start to “drift” in a certain direction towards one of the terminals of the battery. To counter this "drift", one must either disconnect the battery (i.e. block its influence via some insulating operation) or counteract it with an even larger "pull" of voltage.

Even so, spiritually, we must either block the pull of our flesh, or of the enemy, with prayer, fasting, and affirming God’ s Word or yield to a higher pull – that of the Holy Spirit – this is the essence of Romans chapter 8: 3-9 :-

"For the law of the Spirit of life in Christ Jesus hath made me free from the law of sin and death. or what the law could not do, in that it was weak through the flesh, God sending his own Son in the likeness of sinful flesh, and for sin, condemned sin in the flesh: That the righteousness of the law might be fulfilled in us, who walk not after the flesh, but after the Spirit. For they that are after the flesh do mind the things of the flesh; but they that are after the Spirit the things of the Spirit.For to be carnally minded is death; but to be spiritually minded is life and peace. Because the carnal mind is enmity against God: for it is not subject to the law of God, neither indeed can be. So then they that are in the flesh cannot please God. But ye are not in the flesh, but in the Spirit, if so be that the Spirit of God dwell in you..."

It is my wish and prayer in this Lenten season for all of us that we seek and find our way "Home" to God in every way, despite any impurities in us, or any pulls away from Him that we may have within us or encounter without, for He has promised that we shall find Him if we truly seek Him:

"And ye shall seek me, and find me, when ye shall search for me with all your heart." Jer 29:13

I also thank God also that the Good Shepherd goes to seek those who are straying and lost - "For the Son of man is come to seek and to save that which was lost" Luke 19:10 - even when we are not seeking Him:

"As a shepherd seeketh out his flock in the day that he is among his sheep that are scattered; so will I seek out my sheep, and will deliver them out of all places where they have been scattered in the cloudy and dark day." (Ezek 34:12).

When I gave my life to the Lord as a 7 year old child, I had a vision .. I saw in my mind a long white shining straight path, going off into darkness on either side, and I knew it was the path to heaven, and as I prayed with tears, kneeling by myself by my Mother's bedside, and gave my life to Jesus, I prayed that if I strayed from the path that He would put me back on it, and He has honoured that prayer ever since. May we all find Him and not stray from the path He has set before us as we journey Home.

So, let us get drunk, not with wine, wherein is excess, but drunk with the Holy Ghost (Eph 5:18). Let us not be like the despairing drunkard in the flesh who is lost and cannot find his way back home, but let us be despairing drunkards in the Holy Ghost, desperate for the Presence of the Lord in our lives. To Him be all the riches, wisdom, strength, blessing, honour, glory and power for ever and evermore, amen.

PC

NB. Those interested in the science mentioned can also check the following:

1. "Random Walk", general Wikipedia article;
http://en.wikipedia.org/wiki/Random_walk
2. "Random walks and electric networks", P. G. Doyle & J. L. Snell,
http://arxiv.org/PS_cache/math/pdf/0001/0001057.pdf
3. "Application of the lattice Green's functions for calculating the resistance of infinite networks of resistors", J. Cserti,
http://arxiv.org/PS_cache/cond-mat/pdf/9909/9909120.pdf
4. "Perturbation of infinite networks of resistors", J. Cserti, G. David, A. Piroth, http://arxiv.org/PS_cache/cond-mat/pdf/0107/0107362.pdf

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Re reference 2, the following text, which was not circulated with the above message, is a summary of my own ongoing research into random walks and electric networks.

PC

A New General Method of Solving Laplace's Equation using Discrete Harmonic Green's functions

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Abstract

A new general method is presented, based on discrete harmonic Green's (g) functions, for modeling Laplacian potential distribution over homogeneous bounded or unbounded field regions. The method is similar to integral approaches using continuous Green's (G) functions, where potentials need only be computed over the region of interest rather than over the entire field, in contrast to differential finite difference and finite element numerical methods. It computes the g functions, which circumvent the source singularities of the G, by finding the "equivalent" grid spacing and field radius at any given point in discretised space for the Dirichlet problem. Applications abound re the analysis of random walks, electric fields, tri-diagonal matrices, and multiconductor ladder networks.

Keywords

Laplace's Equation, Discrete Harmonic Green's Functions, Dirichlet, Random Walk, Electric Fields, Ladder Networks.

Nomenclature

G(P,Q), g(P,Q) = continuous, discrete Green's functions for points P,Q.
h, h_Q = actual, equivalent grid spacing at Q.
R, R_Q = radius, equivalent radius of field at Q.
r, r_{PQ} = actual, equivalent distance between P and Q.
$HM(a_1, a_2, \ldots a_n)$ = Harmonic Mean of $(a_1, a_2, \ldots a_n) = [(1/n)\Sigma(1/a_i)]^{-1}$
γ = Euler's constant 0.57721566
W = Watson's constant 0.25273100985
K(k) = Legendre's complete elliptical integral of the first kind, modulus k.

$= \int d\theta/\sqrt{[1-k^2 \sin^2\theta]}$

Introduction

In modern times, thanks to computers, numerical methods of field analysis have superseded previous cumbersome analogue methods. The latter included electrical analogues such as resistance networks, conducting paper, and the electrolytic tank. The main numerical approaches now used are probabilistic (using "Monte Carlo" techniques), differential (finite difference and finite element algorithms), and integral. The latter, which utilise Green's functions, are inherently powerful in that they effectively reduce the dimensionality of a problem, requiring only data for field boundaries and sources, while differential methods require information concerning all the field nodes of the discretised field. In practice one may only be interested in finding the potentials of a relatively small section of a given field; integral methods enable one to do his directly without computing any other potentials. The advantage of using discrete harmonic Green's functions (g) to solve Laplace's equation is that one avoids the singularities at the source point of classical Green's functions (G).

For his doctorate, the author developed the following "Finite Difference Boundary Simulation" (FDBS) method. Let $[u_0]$ and $[u_e]$ be the column vector potentials respectively of the original (known) and equivalent (unknown) n boundary potentials of a field region. Then $[u_0]=[g][u_e]$, where [g] is a square symmetric matrix composed of g function values; for its $(i,j)^{th}$ element, g(i,j), i relates to the location of the i^{th} original boundary potential $u_0(i)$, and j to the location of the j^{th} equivalent boundary potential $u_e(j)$. Having inverted [g] to find $[u_e]$, the desired field solution for an arbitrary number of field potentials, say m, is given by $[u] = [g]\,[u_e]$, where [g] is now a (m by n) rectangular matrix; for its $(i,j)^{th}$ element, i relates to the location of the i^{th} field potential u_i, and j to the location of the j^{th} equivalent boundary potential $u_e(j)$. This method can also be converted into an iterative process to minimise required computer storage. Further details are found in (1).

This paper introduces a new approach to the use of the g functions to solve Laplace's equation. Instead of transformation on boundary potentials, a transformation of the field itself is carried out at the nodes of interest, avoiding the need for either matrix inversion or iteration, and enabling a direct solution of the Dirichlet problem (specified boundary potentials). The fewer non-zero boundary potentials present, the faster and more advantageous the method becomes.

The French mathematician Simon de Laplace developed the concept of potential – a function whose directional derivative at every point is equal to the component of field intensity in the given direction. Laplace’s equation $\nabla^2 u = 0$ is of fundamental importance in astronomy and mathematical physics, and governs the behaviour of gravitational, electric and magnetic field potentials. The Swiss mathematician Leonard Euler (1707-1783) studied this equation in 1752 in connection with hydrodynamics, but Laplace made it a standard part of mathematical physics.

For a source or sources within the field domain, Laplace’s equation becomes Poisson’s equation $\nabla^2 u = -f$, where f is a specified function of position. The Dirichlet boundary condition for this equation, where the potential u vanishes on the boundary, is satisfied by the continuous harmonic Green’s function G(P,Q), which gives the response at a field point P due to the input of a Dirac delta function (δ) at a source point Q, when the field potential u_P at P is defined by:

I-1 $u(P) = \int G(P,Q)\, f(Q)\, dQ$

Where $\nabla^2 V_P = -f(P)$

And $\nabla^2 G(P,Q) = -\delta,\ P = Q$

$= 0$, otherwise.

For discretised space, we may denote the discrete harmonic Green’s function which corresponds to G(P,Q) by g(P,Q). Such functions were first discussed by Courant, Friedrichs and Lewy (3). The G and g functions share similar properties; in particular, both are symmetrical, ie. G(P,Q)=G(Q,P), and g(P,Q)=g(Q,P).

In D dimensions, if h_i is a fixed increment in the direction of the i^{th} orthogonal axis, $g(P,Q) \rightarrow G(P,Q)$ as $h_i \rightarrow 0$, and:

I-2 $u(P) = h_1\ldots h_D \, \Sigma \; g(P,Q)\, f(Q)$

Where $\nabla_h^2 u(P) = -f(P)$

$\nabla_h^2 g(P,Q) = -1/(h_1 h_2 \ldots h_n)$, $P = Q$

$= 0$, otherwise.

and ∇_h^2 denotes the finite difference form of the Laplacian operator.

As the distance r between the source point Q and the field point P increases, solutions for g(P,Q) rapidly tend towards those given by G(P,Q). Re bounded fields, exact analytical solutions for G(P,Q) and g(P,Q) can only be derived if the field is a geometrically simple one; solutions for G(P,Q) generally consist of D-order summations to infinity, solutions for g(P,Q) of finite D-order summations. Several examples of both kinds of solutions are to be found in the references.

For an infinite field, g(P,Q) can be expressed as the solution to a D^{th} order integral, eg. for the infinite D-dimensional grid uniformly consisting at every node of $h_1, h_2, \ldots h_D$ increments in orthogonal r_1 to r_D directions:

$$\text{I-3}\quad g(P,Q) = \frac{1}{4\pi^D h_1 h_2 \ldots h_D} \int_0^\pi \int_0^\pi \ldots \int_0^\pi \frac{\cos(x_1 r_1/h_1)\cos(x_2 r_2/h_2)\ldots\cos(x_D r_D/h_D)\,dx_1 \ldots dx_D}{h_1^{-2}\sin^2(x_1/2) + h_2^{-2}\sin^2(x_2/2) + \ldots + h_D^{-2}\sin^2(x_D/2)}$$

Where the distance between P and Q = $\sqrt{(r_1^2 + r_2^2 + \ldots + r_D^2)}$

Whereas G(Q,Q) is indeterminate for any field, g(Q,Q) is always non-infinite for bounded fields, while the infinite field integrals for g(Q,Q) converge for $D \geq 3$, and their singularities in one or two dimensions can be circumvented by defining a function g*(P,Q) such that:

I-4 $g^*(P,Q) = g(Q,Q) - g(P,Q)$

For uniform Laplacian field grid models, N is the number of nodes inside the grid connected to any given node, and h the distance between adjacent nodes. Under these circumstances, Laplace's equation becomes :

I-5 $\nabla^2 u = 2D(S - N u_p)/(Nh^2) = 0$

Where S is the sum of the N neighbouring potentials to a field point P, and u_P their arithmetic average.

Resistance network analogues were successfully used in the past to model Laplacian fields. The resistances between adjacent nodes for uniform grids are equal and given by $Nh^2/(2DA)$, where A is the length/area/volume of the so-called "Dirichlet region" in 1, 2, and 3-D respectively. For a given node, this is the region of space closer to that node than any other, constructed by joining it to neigbouring nodes by straight line segments, bisecting those segments, and finding the smallest convex region bounded by the bisectors. In 1-D, the Dirichlet regions are lines, in 2-D polygons, and in 3-D polyhedra, which fit together to fill space without gaps.

There is a close connection between the g function, resistance networks, Monte Carlo probabilistic field-solving techniques or random walks, and the inversion of tri-diagonal type matrices. When one employs a resistance network analogue for a Laplacian field, earthed at boundary nodes, g(P,Q) corresponds to the voltage at P, and g(Q,Q) to the voltage at Q, due to a 1 Amp input at Q from a source external to the network; if the network is infinite, the electrical resistance between P and Q equals 2g*(P,Q), as noted by Van Der Pol and Bremner [11] for the infinite square grid. The probability p(P,Q) that a random walker on a D-dimensional space-filling grid, starting from Q, and stopping on reaching a boundary, reaches P is given by $\{1-h^2/[2Dh^D g(P,Q)]\}$; for D<3, g(Q,Q) is infinite, and p(Q,Q) =1 (the walker always returns eventually to Q); as $D\rightarrow\infty$, $p(P,Q)\rightarrow 1/(2D)\rightarrow 0$. The column vector of field potentials [u(P)] is given by $[M]^{-1}$ [u(B)], where [M] is a matrix whose order (O) is the number of internal field nodes, and [u(b)] is the column vector of specified boundary potentials. The O^2 values of g(P,Q) are those of the $[M]^{-1}$ matrix; the g(Q,Q) function yields the numbers on its main diagonal.

For example, consider a random walker in the center Q of a wire earthed at both ends with a resistance of 1Ohm per unit length, who generates a current of 1 Amp, and is equally likely to go backwards or forwards after each unit distance travelled, but must stop walking on reaching an end of the wire. If the wire is 2 units long, the walker never returns to Q; if 4 units, the walker has a 1 in 2 chance of returning; if infinite, the walker always returns eventually. In all cases, g(Q,Q) = R/2, the walker's probability of return = 1-0.5/g(Q,Q), the voltage at Q is g(Q,Q), and the voltage at P is g(P,Q). If the walker is removed, and a unit source voltage is applied to one end of the wire, the resulting potential distribution along the wire (a linear function of position) satisfies Laplace's equation in 1 dimension, and P units away equals g(P,1). The corresponding [M] matrix for this problem is a tridiagonal matrix, the number 2 on the main diagonal, and –1 on the adjacent diagonals; its inverse contains the g(P,Q) values, given by X(L-x)/L if the wire is L units long, P being x units, and Q being X units, from the source voltage.

When Q is in the center of a line, circle or sphere of radius R, g(Q,Q) has a value based on R and an "equivalent grid spacing" h_Q , the latter being governed by the geometry of the Dirichlet region associated with Q.

For a general D-dimensional field, g(Q,Q) behaves as though that field's boundary lies separated from Q in all directions by an "equivalent radius" R_Q , where:

I-6 R_Q ~ limit HM [m+m(i)], i = 1,2,…n
as n→:

And:

m = distance between Q and the nearest boundary point (i=1);

i = a point, n in number, on the inscribed line/circle/ sphere centered on Q which touches the boundary, with these n points equi-angularly distributed around Q, and the first point (i=1) being the nearest boundary point to Q.

m(i) = distance between the i^{th} point (on the inscribed line/circle/ sphere centered on Q which touches the boundary) and the nearest (j^{th}) point on the boundary within the half region of space tangential to i which does not include Q.

Note that when i =1, m(1)=0; also, by Pythagoras' Theorem, setting Q to be the center of coordinates, the sum of the products $x_i x_j$, $y_i y_j$, and $z_i z_j$, in 1-D, 2-D, and 3-D respectively, must be greater than or equal to m^2.

The accuracy of I-6 increases as more m(i) points are chosen, and this definition of R_Q allows the handling of both finite and infinite field boundaries. Note that m has the most influence on the value of R_Q.

One may further define the "equivalent distance" r_{PQ} between P and Q as the minimum distance of a series of straight lines which connect P and Q, drawn within the confines of the field from internal node to internal node, without passing within a grid spacing of the field boundary. With this definition, to find g(P,Q) for any given field, bounded or unbounded, one substitutes the function $\sqrt{(r^2_{PQ}+R_P R_Q)}$ for R in the known analytical Green's functions for sources in the center of a line, circle or sphere of radius R, where R_P and R_Q are the equivalent radii of the field at points P and Q respectively.

For a given grid spacing, the method's accuracy can be improved by using 4^{th} order, as opposed to 2^{nd} order, accuracy grids. In 2-D, h_Q=h sec (30^o) for both the 4^{th} order triangular and diagonal grids. The triangular grid's resistance network analogue is composed of √3 ohm resistances. The diagonal grid's resistance network analogue has 1.5 ohm resistances in the x and y directions, and 6 ohm resistances in diagonal directions; the potential at each node is (1/20)[4S1 + S2], where S1 is the sum of potentials at the 4 neighbouring nodes h units away, and S2 is the sum of the potentials at the 4 nodes √2 h units away[1].

For all grids, accuracy can be further improved by using the fact that each potential drop between Q and neigbouring nodes, divided by the intervening grid resistance, sums to unity. In particular, this improves the method's accuracy for nodes a step length away from a field boundary, where accurate computation of R_Q is important.

1.0 1-D ANALYSIS

For a line 2R units in length, with Q positioned r` units from its center, and P r units from Q:

1-1 $g(P,Q) = G(P, Q) = (R-r`)(R+r`-r)/(2R)$

$= [\sqrt{(r^2+R_P R_Q)} – r]/2$

Where R_P and R_Q = HM (distances d_1 and d_2 to the boundary nodes in the +x and –x directions from P and Q respectively.)

When Q is in the center of the line, R_P = HM (R+r, R-r) = $(R^2-r^2)/R$, $R_Q = R$, and $g(P,Q) = G(P,Q) = (R-r)/2$. When $r = r` = 0$, $g(Q,Q) = G(Q,Q) = R/2$. When r=0, and Q is non-centered, $g(Q,Q) = G(Q,Q) = R_Q/2$. Seen from Q, the field therefore behaves at all times as though Q is in the center of a line of radius $R_{Q,}$ the "equivalent radius" of the field at Q; similarly R_P is the field's "equivalent radius" at P. For the infinite line, $g^*(P,Q) = r/2$.

2. 2-D ANALYSIS

For any space-filling uniform infinite 2-D grid:

2-1 $g^*(P,Q) = g(Q,Q)-g(P,Q) = (1/(2\pi))[\gamma + \ln (4 r/h_Q)]$

The appearance of Euler's constant, which has been termed [13] the third most important number in Mathematics after π and e (the base of Naperian logarithms) is of theoretical interest. It is absent in classical theory because it becomes lost in the infinite constant of integration associated with the field of the line charge.

The equivalent grid spacing h_Q is defined as follows. Let hi be the harmonic mean of distances from Q to the vertices of the i^{th} side, length S_i, of the Dirichlet region D_Q, a polygon with perimeter S whose N sides bisect the nearest grid spacings around Q. Then h_Q is double the HM of the h_i variables, each weighed by S_i, ie. $h_Q = (2N/S)HM(S_i h_i)$

As D_Q becomes more circular, $h_Q \rightarrow h$, the minimum distance from Q to a node. For uniform space-filling 2-D grids, $h_Q = h \sec(\pi/N) = 2h$ for the hexagonal grid, $h\sqrt{2}$ for the square grid, and $2h/\sqrt{3}$ for the triangular grid. Corresponding analogue resistances are each $\cot(\pi/N)$, when the potential drop from Q to the nearest nodes is $(1/N)\cot(\pi/N)$, with which 2-1 roughly agrees for r=h. For the rectangular grid, with spacings h_x and h_y in the x and y directions respectively between nodes, g*(P,Q) is the sum of the reciprocals of the first n odd numbers, divided by π, where r is $n\sqrt{(h_x^2+h_y^2)}$, with which 2-1 also agrees.

One check of 2-1 is to quantise Gauss' law re an infinite line charge. Let Gauss' law be valid only over cylinders of unit height at radii h/2, 3h/2, 5h/2 etc from a line charge of 1 Coulomb/unit length. Applying the law over the smallest cylinder, we find the electric field strength E at its surface is the reciprocal of its curved surface area (πh). Hence the potential drop from the line charge to a distance h units radially away is E h = 1/π. Similarly applying Gauss' law over the cylinder with radius 3h/2, we find the second potential drop, from r=h to r=2h, to be 1/(3π), etc. The potential drop S summed to distance nh units are the sum of the reciprocal of the first n odd numbers, divided by π. Since the sum S_i of the reciprocal of the first i natural numbers ~ (γ+ ln i), and $S_{2i} = S + □ S_i$, 2-1 follows for $h_Q=h$. Another way of obtaining the same result is to study the electrical network grid analogue for Laplace's equation in plane polar coordinates. Here radial resistances are given by $h_r/[(r6h_r/2)h_υ]$ corresponding to the radial increment $6h_r$, and angular ones by $rh_υ/h_r$ corresponding to the angular arc $6h_υ$. If unit current is injected into the origin, current flow will divide up $(2\pi/h_υ)$ ways and be entirely radial; g(Q,Q) will then be the potential at the origin after a current of $(h_υ/2\pi)$ Amps has travelled through the series radial resistances $2/h_υ$, $2/(3h_υ)$, $2/(5h_υ)$…etc. Again, 2-1 follows where $h_Q=h_r$. If Q is not at the origin of coordinates, then h_Q is $\sqrt{[h_r^2+(rh_θ)^2]}$.

Any point C on the circumference of a circle radius R, center O, is such that, for any internal point Q radius A from O, the ratio of the lines QC to Q`C is constant, where Q` is the image or inverse point of Q and is positioned outside the circle at a radial distance (R^2/A) from its center, on the extension of the line OQ. Using this fact, it may be shown that for the circle:

2-2 $G(P,Q) = (1/(2\pi)) \ln (A\, r`/R\, r)$, $A > 0$, $r>0$
$= (1/(2\pi)) \ln(R/r)$, $A= 0$, $r>0$
Where r (r`)= distance between points P and Q (Q`).

For the general arbitrarily-shaped 2-D field:

2-3 $g(P,Q) \sim (1/(2\pi)) \ln [\sqrt{(r^2_{PQ} + R_P R_Q)}/r]$ $r>0$
$\sim (1/(2\pi)) \{\gamma + \ln (4 R_Q/h_Q)\}$, $r=0$

Where:

R_P , R_Q = the field's equivalent radii at P and Q respectively;

r_{PQ} = distance of shortest path between P and Q that does not come within distance h of the boundary of the field.

If the field is circular, radius R, and Q is in the center, $R_Q = R$, and $R_P = (R^2 - r^2)/R$, when 2-3 agrees with 2-2; here R_P is the HM of the diameters of n circles (n >1) symmetrically arrayed around Q, circumferences tangential to the field boundary, intersecting at Q.

<u>3. 3-D ANALYSIS</u>

In 3-D, the infinite grid resistance network analogue for Laplace's equation models the way gravitational, electric, or magnetic field potential spreads from a source at Q in open space; the grid voltages model potential, the currents model flux. If the N nearest nodes to Q are equidistant, distance h away, and R_N is the grid resistance between Q and any of these nodes, then:

$3\text{-}1 \quad g(Q,Q) \sim (1/(4\pi h)) + R_N/N$

The Green's function for infinite space is given by $G(P,Q) = 1/(4\pi r)$, where r is the distance between P and Q. For the sphere radius R where Q is distance A from its center O:

$3\text{-}2 \quad G(P,Q) = (1/(4\pi))\{(1/r) – (R/Ar`))\}$

Where r` = distance between P and Q`, the image point to Q located distance (R^2/A) from O.

For the infinite discretised grid:

$$3\text{-}3 \quad g(P,Q) \sim G(P,Q) \sim 1/(4\pi r)\ ,\ r>0$$
$$\sim \pi/(8\ h_Q),\ r=0$$

Here h_Q is 2 HM (h_i variables, each weighed by the associated surface area of the i^{th} face of D_Q), where h_i = HM (distances from Q to all the midpoints and vertices of the i^{th} face of D_Q). As D_Q becomes more spherical in shape, $h_Q \rightarrow h$; this can be checked by allowing Gauss' law to be valid at spherical surfaces with radii r=h/2, 3h/2, 5h/2 etc. around a unit point charge Q, when the nearest potential drop to Q is $1/(\pi h)$, followed by $1/(9\pi h)$, etc; and the potential drop from Q to a point distance (nh) units away is the sum S of the reciprocal of the square of the first n odd numbers, divided by πh; as $n \rightarrow \infty$, $S \rightarrow \pi^2/(8h)$. Theoretical Physics presently allows the existence of infinite zero-point energies, but maybe at the sub-atomic level Gauss' law really should be quantised, imposing limits on energy levels.

The resistance network analogue for fields in spherical coordinates has radial resistances $h_r/[(r6h_r/2)^2 h_\upsilon \sin\upsilon\, h_\varphi]$, and angular resistances $\sin\upsilon\, h_\varphi/(h_r h_\theta)]$ and $h_\upsilon/[h_r \sin(\upsilon+h_\upsilon/2)h_\varphi]$, which, from the general point (r,υ,φ), span the radial increment 6hr, and the angular arcs h_υ and h_φ respectively. For the infinite field, current flow from a 1 Amp input at the origin will be entirely radial, and the network potentials will possess spherical symmetry, when, for Q at the origin:

$$g(P,Q) = [1/(h_r h_\theta)] \tan(h_\theta/2)\{(\pi/4) - [1/(2\pi)] \sum_{m=1\ldots r/h_r}[1/(m-½)]^2\}$$
$$\rightarrow 1/(4\pi r) \text{ as } h_\theta \rightarrow 0 , r>0$$
$$= [1/(2\pi)] \sum_{m=1\ldots\infty} [1/(m-½)]^2 \{h_r h_\theta \sum_{n=1\ldots((\pi/h_\theta)-1} \sin(nh_\theta)\}^{-1} = [\pi \tan (h_\upsilon/2)]/(4 h_r h_\upsilon)$$
$$\rightarrow \pi/(8h_r) \text{ as } h_\theta\rightarrow 0 , r=0$$

For the general arbitrarily-shaped 3-D field:

$$\text{3-4 } g(P,Q) = (1/(4\pi))[(1/r) - 1/\sqrt{(r^2+R_P R_{Q)}}], r>0$$
$$= [\pi/(8h_Q)] - [1/(4\pi R_Q)], r=0$$

Where R_P (R_Q) is the equivalent radius of the field at P (Q), and r = distance of shortest path between P and Q within the confines of the field boundary which does not come within h units of that boundary. If the field is spherical, radius R, with Q at its center, then $R_P = (R^2-r^2)/R$, and $R_Q = R$, when 3-4 agrees with 3-2 for r>0.

Table 1 compares values of g(Q,Q) (to 5DP for uniform space-filling lattice structures found in nature) with those given by 3-1 and 3-3; SC = Simple Cubic, BCC = Body-Centred-Cubic, FCC= Face-centred-Cubic. The diamond grid is the carbon-hydrogen structure found in diamonds. Re D_Q, as the Russian crystallographer E. S. Fedorov (1853-1919) proved, all space-filling convex polyhedra can be classified into 5 topological types: the cube, the hexagonal prism, the rhombic dodecahedron, the truncated octahedron, and Fedorov's "elongated dodecahedron" with 8 rhombic and 4 hexagonal faces.

Table 1

N	Type of grid (infinite)	D_Q	h R_N	g(Q,Q) actual	g(Q,Q) by 3-1	g(Q,Q) by 3-3
4	Diamond	Hexagonal Prism	√3/4	0.19409	0.18783	0.19451
6	S.C. grid	Cube	1	0.25273 =W	0.24624	0.25220
8	B.C.C. grid	Truncated Octahedron	√3	0.30164	0.29608	0.30304
12	F.C.C grid	Rhombic Dodecahedron	2√2	0.31694	0.31528	0.31703
∞	Spherical	Sphere	1/π	0.3927	0.3979	0.3927

The triple integral associated with g(Q,Q) for the SC grid (cf. I-3) was solved in 1939 by the British mathematician G.N Watson [12], its value will be referred to as Watson's constant W, whose paper also furnished the solutions for the triple integrals associated with g(Q,Q) for the BCC and FCC grids, ie:

$g_{SC}(Q,Q) = (0.5/h)((18+12\sqrt{2}-10\sqrt{3}-7\sqrt{6})\{(2/\pi)K[\{2-\sqrt{3})(\sqrt{3}-\sqrt{2}]\}^2$

$g_{BCC}(Q,Q) = \sqrt{3}K_0^2/(2h\pi^2)$ where $K_0 = K(\sin 45^o)$

$g_{FCC}(Q,Q) \sim \sqrt{3}\,K_1^2/(2h\pi^2)$ where $K_1 = K(\sin 15^o)$

In general, $g_i(Q,Q) / g_j(Q,Q) \sim R_N^i / R_N^j$ where i relates to D_Q, and j to the polyhedron reciprocal to that outlined by the Q's N neighbouring nodes when symmetrically arrayed around Q, its faces bisecting lines from Q to those nodes. For the SC lattice, i is a cube, neigbouring nodes are already symmetrically arrayed around Q, outlining an octahedron, and j is the same cube as i. For the BCC lattice, i is a truncated octahedron (14 faces: 8 hexagons, 6 squares), Q's 8 neigbour nodes symmetrically outline a cube, and j is a octahedron with resistances 8/(3h√3), for which g(Q,Q)~0.27203 by 3-3. For the FCC lattice, i is a rhombic dodecahedron; Q's 12 neighbour nodes when symmetrically arrayed around Q outline an icosahedron, j is a dodecahedron where R_N = [24/(5h)]tan(π/5), and g(Q,Q) ~ 0.32324. For the diamond grid, i is an hexagonal prism, j is a tetrahedron ; a 3-D delta-star transformation on the resistances of the FCC grid yields the diamond's, the grid spacing of the FCC being (2√2/√3) greater.

The Ring Charge

The Green's function associated with the ring charge is important for solving fields with axial symmetry in cylindrical circular coordinates.

For a point charge at the origin, where $h_r = h_z$ = h units, the author has been able to show by computer that g(Q,Q) ~ 0.2952. This checks with 3-3; D_Q is a multi-sided prism tending towards a cylinder of height and diameter h units, when h_Q is … and g(Q,Q) ~ 0.3. Here g(P,Q) → 1/(4πr) as r→∞.

The continuous Green's function associated with the ring charge is given by:

$$G(P,Q) = (1/(2\pi))\,[\sqrt{(r`/r)}]kK(k)$$

For which $k = 2[\sqrt{(r\,r`)}]/R_2$ , and $k` = 1-k^2 = R_1/R_2$,
Where $R_1 = r[(r-r`)^2 + (z-z`)^2]$ and $R_2 = r[(r+r`)^2 + (z-z`)^2]$

Now it is known that K(k) → ln(4/k`) as k→1, where $(k`)^2 = 1-k^2 = R_1/R_2$, which occurs for the field close to the ring charge, and as r`→∞, this field region behaves like that close to an infinite line charge. It follows that:

$$g(Q,Q) \rightarrow (1/(2\pi))[\gamma + \ln(32\,r/h_Q)] \text{ as } r` \rightarrow \infty.$$

Where $h_Q = \sqrt{(h_r^2 + h_z^2)}$

The author has verified this result experimentally. Note the reappearance of Euler's constant.

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